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Journal of
APPLIED
MATHEMATICS
AND MECHANICS
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Journal of Applied Mathematics and Mechanics 69 (2005) 875–884

CONSTRUCTION OF THE VIABILITY KERNEL FOR A NON-LINEAR CONTROLLED SYSTEM WITH A TARGET SET†

A. A. NEZNAKHIN and V. N. USHAKOV
Moscow and Ekaterinburg
email: a.neznakhin@mail.ru

(*Received* 29 *December* 2004)

A non-linear controlled system in a finite time interval with phase constraints and a given target set is considered. The problem of constructing the viability kernel in the phase constraints is investigated. The viability kernel is the set of all initial positions from which at least one viable trajectory emerges, that is, the trajectory of the system constrained by the phase constraints and which enters the target set. A method of constructing time-discrete approximations of the viability kernel is proposed.

This paper touches on the investigations in [1–10]‡ into viability theory and the theory of differential games. In the development of the problem of constructing a viability kernel, a target set inside the phase constraints is introduced. Here, a trajectory which remains within the phase constraints right up to its arrival in the target set is considered to be viable.

1. FORMULATION OF THE PROBLEM

Suppose a controlled system is specified, the behaviour of which in the time interval $\mathcal{I} = [t_0, \theta]$ $(t_0 \le \theta < \infty)$ is described by the equation

$$\dot{x} = f(t, x, u), \quad x[t_0] = x_0, \quad u \in \mathcal{P} \tag{1.1}$$

Here x is an m-dimensional phase vector from the Euclidean space $\mathbb{R}^m$, u is a control and $\mathcal{P}$ is a compactum in the Euclidean space $\mathbb{R}^p$.

It is assumed that the following conditions are satisfied.

1. The vector function $f(t, x, u)$ is defined and continuous by totality of all its arguments in the set $\mathcal{I} \times \mathbb{R}^m \times \mathcal{P}$ and, for a certain $\mu \in (0, \infty)$, satisfies the inequality

$$\|f(t, x, u)\| \le \mu(1 + \|x\|), \quad \forall (t, x, u) \in \mathcal{I} \times \mathbb{R}^m \times \mathcal{P}$$

$\|x\|$ is the Euclidean norm of the vector $x \in \mathbb{R}^m$.

2. The vector function $f(t, x, u)$ is locally Lipschitzian with respect to x and, for any compactum $\mathcal{D} \subset \mathbb{R}^m$, a constant $L = L(\mathcal{D}) \in (0, \infty)$ exists such that

$$\|f(t, x_1, u) - f(t, x_2, u)\| \le L\|x_1 - x_2\|, \quad \forall (t, x_1, u), (t, x_2, u) \in \mathcal{I} \times \mathcal{D} \times \mathcal{P}$$

By permissible controls of system (1.1), we mean any Lebesgue measurable vector-function $u[t]$, $t \in \mathcal{I}$ which takes values from $\mathcal{P}$ almost everywhere in $\mathcal{I}$.

†*Prikl. Mat. Mekh.* Vol. 69, No. 6, pp. 976–985, 2005.

‡ PATSKO, V. S., and TUROVA, V. L., Numerical solution of differential games in a plane. Preprint, UrO Ross. Akad. Nauk, Ekaterinburg, 1995.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.11.009

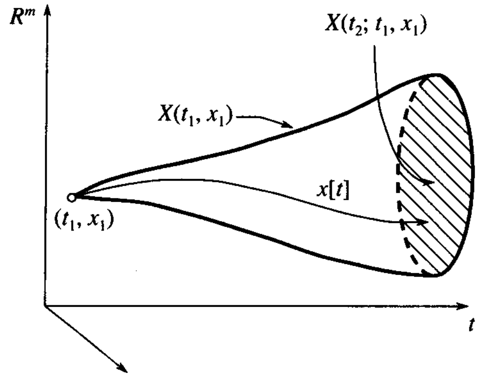

Fig. 1

Suppose $u[t]$, $t \in \mathcal{I}$ is a certain permissible control. We shall call the absolutely continuous vector-function $x[t]$, $t \in \mathcal{I}$, which takes values in $\mathbb{R}^m$ and satisfies the equality

$$\dot{x}[t] = f(t, x[t], u[t]) \text{ almost everywhere in } \mathcal{I}$$

the trajectory of controlled system, which is generated by the permissible control $u[t]$.

The trajectory of controlled system(1.1), which is defined in the interval contained in $\mathcal{I}$, is defined in precisely the same way.

The totality of the trajectories of system (1.1), defined in the interval $[t_1, \theta]$ and generated by all possible permissible controls $u[t]$, $t \in [t_1, \theta]$ such that $x[t_1] = x_1$, is denoted by the symbol $X(t_1, x_1)$, where $(t_1, x_1) \in \mathcal{I} \times \mathbb{R}^m$. We assume that $X(t_2; t_1, x_1) = \{x[t_2]: x[t] \in X(t_1, x_1)\}$, $(t_1, x_1) \in \mathcal{I} \times \mathbb{R}^m$, $t_2 \in [t_1, \theta]$ is the attainability set of system (1.1) at the instant t_2 (see Fig. 1).

We will assume that a phase constraint is specified for system (1.1), that is, a closed set $\Phi \subset \mathcal{I} \times \mathbb{R}^m$ having non-empty sections $\Phi(t) = \{x \in \mathbb{R}^m: (t, x) \in \Phi\}$, $t \in \mathcal{I}$.

A definition of a viable trajectory $x[t]$, $t \in [t_1, \theta]$ of a controlled system has been given in [9] as a trajectory which satisfies the condition $x[t] \in \Phi(t)$, $t \in [t_1, \theta]$.

We will now consider the case when, together with the set Φ, a constraint and a target set $T \subset \Phi$, $T(\theta) \neq \varnothing$, which is closed in $\mathcal{I} \times \mathbb{R}^m$, are specified.

Definition 1. We will call a trajectory $x[t]$ of system (1.1), which is defined in the interval $I = [t_1, t_2] \subset \mathcal{I}$, a viable trajectory (VT) if the condition $x[t] \in \Phi(t)$, $t \in I$; $x[t_2] \in T(t_2)$ is satisfied, where t_2 is the minimum instant at which the trajectory $x[t]$ enters the target set T.

A viable trajectory is shown schematically in Fig. 2.

Note the connection between the definition of a target set from [9] and the definition which has been presented here. Suppose the target set is defined by the relations $T(t) = \varnothing$, $t \in [t_0, \theta)$; $T(\theta) = \Phi(\theta)$. The trajectory $x[t]$, $t \in [t_1, \theta]$ of controlled system (1.1) is a viable trajectory in the sense of Definition 1 if and only if it is a viable trajectory in the sense of the definition from [9].

The condition for the target set T to be bounded guarantees that a certain compactum $\mathcal{D}' \subset \mathbb{R}^m$ exists such that $T \subset \mathcal{I} \times \mathcal{D}'$. The following lemma then follows from condition 1.

Lemma 1. A compactum $\mathcal{D} \subset \mathbb{R}^m$ exists which contains the phase portraits of all the viable trajectories $x[t]$, $t \in I$ of controlled system (1.1): $x[t] \in \mathcal{D}$, $t \in I$.

We will now formulate a fundamental definition.

Definition 2. We call the set of all points $(t_1, x_1) \in \mathcal{I} \times \mathbb{R}^m$, from which at least one viable trajectory of system (1.1) emerges the *viability kernel* Ω' of system (1.1).

The viability kernel is shown schematically in Fig. 3.

The following lemma follows from Lemma 1.

Lemma 2. A compactum $\mathcal{D} \subset \mathbb{R}^m$ exists such that $\Omega' \subset \mathcal{I} \times \mathcal{D}$.

Assertion 1. $T \subset \Omega' \subset \Phi$.

The aim of this paper is to indicate a method for constructing the kernel Ω'. Here, we replace controlled system (1.1) with a differential inclusion which corresponds to system (1.1) and enables one

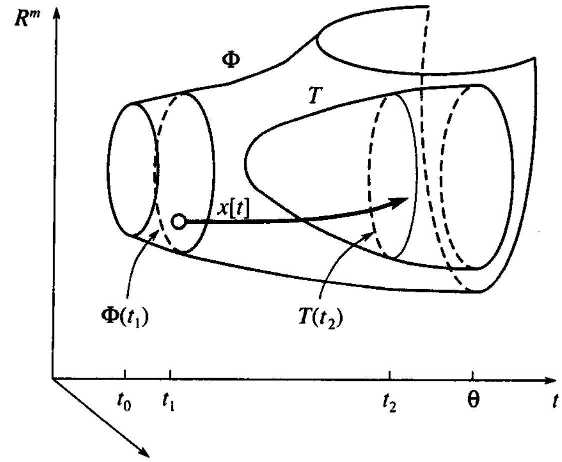

Fig. 2

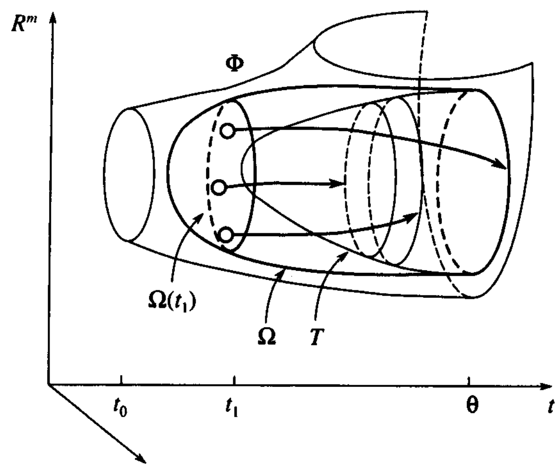

Fig. 3

to apply the technique of approximate calculations of stable bridges, which is used in the theory of positional differential games (see [10], for example), to the problem of constructing the viability kernel.

Instead of controlled system (1.1), we consider the differential inclusion

$$\dot{x} \in F(t, x), \quad (t, x) \in \mathcal{I} \times \mathbb{R}^m \tag{1.2}$$

where $F(t, x)$ is the convex hull of the set $\{f(t, x, u)\colon u \in \mathcal{P}\}$.

Inclusion (1.2) satisfies the following conditions, which are analogous to conditions 1 and 2, imposed on system (1.1).

1. The set-valued map $(t, x) \to F(t, x)$ is continuous by totality of (t, x) in the Hausdorff metric and a constant $\mu \in (0, \infty)$ exists such that the following condition is satisfied

$$\sup_{f \in F(t,x)} \|f\| \le \mu(1 + \|x\|), \quad \forall (t, x) \in \mathcal{I} \times \mathbb{R}^m$$

2. The set-value map $(t, x) \to F(t, x)$ is locally Lipschitzian with respect to x in the Hausdorff metric, that is, a constant $L = L(\mathcal{D}) \in (0, \infty)$ is found for any compactum $\mathcal{D} \subset \mathbb{R}^m$ such that the condition

$$d(F(t, x_1), F(t, x_2)) \le L\|x_1 - x_2\|, \quad \forall (t, x_1), (t, x_2) \in \mathcal{I} \times \mathcal{D}$$

for the Hausdorff distance is satisfied.

We shall call an absolutely continuous vector function $y[t]$, $t \in I \subset \mathcal{I}$, which takes values in $\mathbb{R}^m$ and which satisfies the inclusion

$$\dot{y}[t] \in F(t, y[t]) \text{ almost everywhere in } I$$

a trajectory of inclusion (1.2).

In the same way as above in the case of inclusion (1.2), we will now introduce into the treatment the concept of the set of all trajectories which are defined in the interval $[t_1, \theta]$ and emerge at the instant t_1 from the point x_1: $Y(t_1, x_1)$, $(t_1, x_1) \in \mathcal{I} \times \mathbb{R}^m$ and the concept of an attainability set: $Y(t_2; t_1, x_1) = \{y[t_2]: y[t] \in Y(t_1, x_1)\}$.

We will now also give a definition of a viable trajectory.

Definition 3. We will call a trajectory $y[t]$ of inclusion (1.2), defined in the interval $[t_1, t_2] \subset \mathcal{I}$, a *viable trajectory* if the following conditions are satisfied

$$y[t] \in \Phi(t), \quad t \in [t_1, t_2]; \quad y[t_2] \in T(t_2); \quad y[t] \notin T(t), \quad t \in [t_1, t_2) \tag{1.3}$$

Like to Lemma 1, the following lemma is true.

Lemma 3. A compactum $\mathcal{D} \subset \mathbb{R}^m$ exists such that $y[t] \in \mathcal{D}$, $t \in [t_1, t_2]$.

Definition 4. We shall call the set Ω of all the points $(t_1, x_1) \in \mathcal{I} \times \mathbb{R}^m$ from which at least one viable trajectory of the inclusion (1.2) emerges the *viability kernel* of inclusion (1.2).

Lemma 4. A compactum $\mathcal{D} \subset \mathbb{R}^m$ exists such that $\Omega \subset \mathcal{I} \times \mathcal{D}$.

Like Assertion 1, the following is true.

Assertion 2. $T \subset \Omega \subset \Phi$.

The kernel Ω is close to the kernel Ω' and their relation is characterized by the closeness of the attainability sets: the sets $Y(t_2; t_1, x_1)$ is equal to the closure of the set $X(t_2; t_1, x_1)$.

2. A TIME-DISCRETE APPROXIMATION OF THE VIABILITY KERNEL Ω

We now propose a method for the approximate construction of the set Ω, using constructions similar to those described earlier in [9]. Here, the set $Y^{-1}(t_1; t_2, x_2) = \{x_1 \in \mathbb{R}^m: x_2 \in Y(t_2; t_1, x_1)\} \subset \mathbb{R}^m$ of all the points x_1 at which the trajectories of the inclusion

$$\dot{y}[\tau] \in F^*(\tau, y), \quad y[t_1] = x_2; \quad F^*(\tau, y) = -F(t_1 + t_2 - \tau, y), \quad \tau \in I = [t_1, t_2]$$

arrive at the instant $\tau = t_2$ plays an important role.

This inclusion can be treated as inclusion (1.2) defined in terms of an "inverse" time τ. We now establish the correspondence of the set

$$\tilde{Y}^{-1}(t_1; t_2, x_2) = x_2 - (t_2 - t_1)F(t_2, x_2) \subset \mathbb{R}^m$$

to the set $Y^{-1}(t_1; t_2, x_2)$.

The former set depends on $(t_2 - t_1)$ of the length of the segment I and approximates the set $Y^{-1}(t_1; t_2, x_2)$ well. In fact, the following estimate holds

$$d(Y^{-1}(t_1; t_2, x_2), \tilde{Y}^{-1}(t_1; t_2, x_2)) \le \omega(t_2 - t_1); \quad \lim \frac{\omega(t_2 - t_1)}{t_2 - t_1} = 0 \quad \text{when} \quad t_2 - t_1 \to 0$$

where the scalar function $\omega(t_2 - t_1)$ is independent of the point (t_2, x_2) and satisfies the above-mentioned limit equality.

We also define the set

$$\tilde{Y}^{-1}(t_1; t_2, X_2) = \bigcup\{\tilde{Y}^{-1}(t_1; t_2, x_2): x_2 \in X_2\} \subset \mathbb{R}^m$$

We now turn to the basic constructions. For each natural n, we introduce a partitioning $\Gamma_n = \{t_0 = t^{(0)}, t^{(1)}, t^{(2)}, \dots, t^{(n)} = \theta\}$ of the interval $\mathcal{I}$ which is uniform with instants $t^{(i)} \in \mathcal{I}$ and a diameter

$$\Delta_n = (\theta - t_n)/n, \quad t^{(i)} - t^{(i-1)} = \Delta_n, \quad i = 1, 2, \dots, n$$

We now fix a certain n. The partitioning Γ_n corresponds to it. We make the target set T discrete in time by associating with it the set T_n of the sets $T_n(t^{(i)}) \subset \mathcal{D}$, $t^{(i)} \in \Gamma_n$, defined by the equalities

$$T_n(t^{(0)}) = \varnothing, \quad T_n(t^{(i)}) = \bigcup\{T(t): t \in [t^{(i-1)}, t^{(i)}]\}, \quad i = 1, 2, \dots, n$$

We will call this set discretization of the target set T.

We let

$$\begin{aligned} K &= \max\{\|f(t, x, u)\|: (t, x, u) \in \mathcal{I} \times \mathcal{D} \times \mathcal{P}\} < \infty \\ \omega(\Delta) &= \Delta\omega^*((1+K)\Delta), \quad \Delta > 0 \\ \omega^*(\Delta) &= \sup\{d(F(t_1, x_1), F(t_2, x_2)): (t_1, x_1), (t_2, x_2) \in \mathcal{I} \times \mathcal{D}, |t_1 - t_2| + \|x_1 - x_2\| \le \Delta\} \end{aligned} \tag{2.1}$$

Here, $\omega^*(\Delta)$ is the modulus of the continuity of the map $(t, x) \to F(t, x)$.

We now recurrently assign the numbers $\varepsilon^{(i)} \ge 0$ ($i = n, n-1, \dots, 0$), associated with the instants $t^{(i)}$ of the partitioning Γ_n as follows:

$$\varepsilon^{(n)} = K\Delta_n; \quad \varepsilon^{(i)} = \omega(\Delta_n) + (1 + L\Delta_n)\varepsilon^{(i+1)}, \quad i = n-1, n-2, \dots, 0$$

Assertion 3.

$$\lim \max_i \varepsilon^{(i)} = 0$$

Here and henceforth the limit is taken as $n \to \infty$.

The proof is similar to the proof of a similar assertion in [9].

In the case of fixed n, we shall approximate the viability kernel Ω with the collection of sets

$$\{\Omega_n(t^{(i)}) \subset \mathcal{D}: t^{(i)} \in \Gamma_n\} \tag{2.2}$$

which we determine recurrently, starting from the last instant $t^{(n)} = \theta$ and finishing with the first instant $t^{(0)} = t_0$.

Definition 5. We will call the collection of sets (2.2), defined by the rules $\Omega_n(t^{(n)}) = T_n(t^{(n)})$, the discrete approximation of the kernel Ω; $\Omega_n(t^{(i)})$ ($i = n-1, n-2, \dots, 0$) is determined in three stages: (1) $\bar{\bar{\Omega}}_n(t^{(i)}) = \tilde{Y}^{-1}(t^{(i)}; t^{(i+1)}, \Omega_n(t^{(i+1)}))$, (2) $\bar{\Omega}_n(t^{(i)}) = \bar{\bar{\Omega}}_n(t^{(i)}) \cap \Phi(t^{(i)})_{\varepsilon^{(i)}}$, (3) $\Omega_n(t^{(i)}) = \bar{\Omega}_n(t^{(i)}) \cup T_n(t^{(i)})$.

Here and below, X_ε is the closure of the ε-neighbourhood of the set $X \subset \mathbb{R}^m$.

From the discrete approximation of the kernel Ω (see Fig. 4), we change to its limit approximation.

Definition 6. We will call the set Ω_0 of all the points $(t^0, x^0) \subset \mathcal{I} \times \mathcal{D}$ which can be represented in the form

$$(t^0, x^0) = \lim(t^n, x^n), \quad t^n \ge t^0; \quad (t^n, x^n) \in \Gamma_n \times \Omega_n(t^n), \quad n = 1, 2, \dots \tag{2.3}$$

the limit approximation of the kernel Ω.

The inclusion $T \subset \Omega^0 \subset \Phi$ holds.

3. FUNDAMENTAL THEOREM

We will now formulate a basic assertion which justifies the introduction of the discrete approximations $\Omega_n(t^{(i)})$.

Theorem.

$$\Omega^0 = \Omega$$

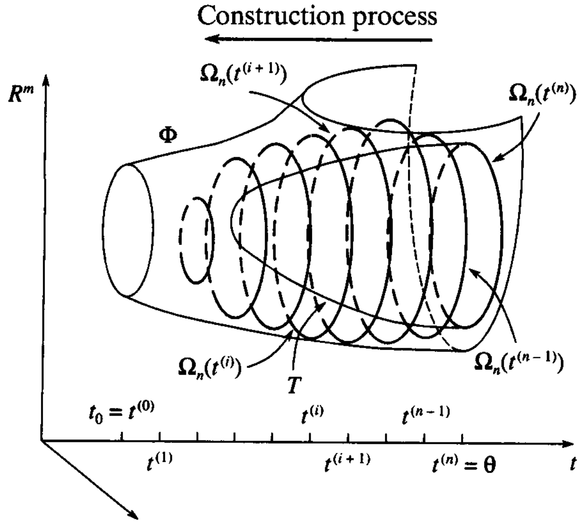

Fig. 4

Proof. We will first prove the inclusion $\Omega^0 \subset \Omega$ and, then, the inclusion $\Omega \subset \Omega^0$.

To prove the inclusion $\Omega^0 \subset \Omega$, we will select an arbitrary point $(t^0, x^0) \in \Omega^0$ and show that $(t^0, x^0) \in \Omega$. Using the definition of Ω^0, the sequence of points (2.3) is found.

We fix an arbitrary number n and consider the point $(t^n, x^n) \in \Omega_n$. Using the definition of $\Omega_n(t^n)$, the Euler's broken line $\tilde{y}_n[t]$ is found, which is defined in the interval $[t^n, t_n]$, $t_n \geq t^n$, $t_n \in \Gamma_n$ and linearly constant in the intervals $(t^{(i)}, t^{(i+1)}) \subset [t^n, t_n]$ of the partitioning Γ_n:

$$\dot{\tilde{y}}_n[t] = \text{const} \in F(t^{(i+1)}, \tilde{y}_n[t^{(i+1)}]), \quad t \in (t^{(i)}, t^{(i+1)})$$

The following conditions are satisfied here.

1. At the instant t^n, the broken line $\tilde{y}_n[t]$ emerges from the point x^n:

$$\tilde{y}_n[t^n] = x^n \tag{3.1}$$

and, at the instants $t^{(i)} \in [t^n, t_n]$ of the partitioning, Γ_n satisfies the inclusion $\tilde{y}_n[t^{(i)}] \in \Phi(t^{(i)})_{\varepsilon^{(i)}}$

2. At the instant t_n, the broken line $\tilde{y}_n[t]$ enters the set T_n:

$$\tilde{y}_n[t_n] \in T_n(t_n) \tag{3.2}$$

3. The instant $t^{(i)} = t_n$ is the minimum instant of the partitioning Γ_n, at which the broken line $\tilde{y}_n[t]$ enters T_n:

$$\tilde{y}_n[t^{(i)}] \notin T_n(t^{(i)}), \quad t^{(i)} \in [t^n, t_n), \quad t^{(i)} \in \Gamma_n \tag{3.3}$$

Suppose $t_n^{(i)} = t_n - \Delta_n \in \Gamma_n$ and consider the interval $[t_n^{(i)}, t_n]$. It follows from the inclusion (3.2), the definition of the set T_n and definition (2.1) that an instant $\tau_n \in [t_n^{(i)}, t_n]$ can be found at which the following inclusion holds

$$\tilde{y}_n[\tau_n] \in T(\tau_n)_{K\Delta_n} \tag{3.4}$$

In $T(\tau_n)$, we select the point y_n which is closest to $\tilde{y}_n[\tau_n]$. The following inequality is satisfied

$$\|y_n - \tilde{y}_n[\tau_n]\| \leq K\Delta_n \tag{3.5}$$

Hence, Euler's broken line $\tilde{y}_n[t]$, $t \in [t^n, t_n]$ is determined for each n, which satisfies conditions (3.1)–(3.3), and the instant τ_n is determined at which inclusion (3.4) is satisfied. Without loss of generality, we shall assume that the following limit exists

$$\lim \tau_n = \bar{\tau} \tag{3.6}$$

We now extend the definition of all the functions $\tilde{y}_n[t]$, $t \in [t^n, t_n]$ to the interval $I^0 = [t^0, \bar{\tau}]$ and we establish a correspondence of the function $\tilde{y}_n[t]$ to each broken line $\tilde{x}_n[t]$, $t \in I^0$:

$$\tilde{x}_n[t] = \begin{cases} \tilde{y}_n[t^n], & t \in [t^0, t^n] \\ \tilde{y}_n[t], & t \in (t^n, \tau_n), \\ \tilde{y}_n[\tau_n], & t \in [\tau_n, \bar{\tau}] \end{cases} \quad \tau_n < \bar{\tau}; \quad \tilde{x}_n[t] = \begin{cases} \tilde{y}_n[t^n], & t \in [t^0, t^n] \\ \tilde{y}_n[t], & t \in (t^n, \bar{\tau}) \end{cases}, \quad \tau_n \geq \bar{\tau}$$

From the uniformly bounded and equicontinuous sequence $\{\tilde{x}_n[t]\}$ in the interval I^0, we separate out a uniformly converging subsequence. Without loss in generality, we shall assume that the sequence $\{\tilde{x}_n[t]\}$ converges itself uniformly and suppose that $x[t] = \lim \tilde{x}_n[t]$, $t \in I^0$.

Using standard techniques, it can be shown that the function $x[t]$ is a trajectory of inclusion (1.2), which does not leave the constraint Φ over the whole of the interval I^0. We will now show that $x[t^0] = x^0$. Actually, from the definition of the functions $x[t]$ and $\tilde{x}_n[t]$, we have

$$x[t^0] = \lim \tilde{x}_n[t^0] = \lim \tilde{y}_n[t^n]$$

from which, when account is taken of condition (3.1) and the equality $\lim x^n = x^0$, we obtain $x[t^0] = x^0$.

We will show that, all the instant $t = \bar{\tau}$, the trajectory $x[t]$ enters the target set. In fact, in a similar manner to the above, we have

$$x[\bar{\tau}] = \lim \tilde{x}_n[\bar{\tau}] = \lim \tilde{y}_n[\tau_n] \tag{3.7}$$

(when the convergence (3.6) is used in writing the last equality) It follows from inequality (3.5) that

$$\lim \tilde{y}_n[\tau_n] = \lim y_n \tag{3.8}$$

Since $y_n \in T(\tau_n)$, by virtue of the convergence (3.6) and the target set to be closed, we obtain

$$\lim y_n \in T(\bar{\tau}) \tag{3.9}$$

It follows from the limit relations (3.7)–(3.9) that $x[\bar{\tau}] \in T(\bar{\tau})$.

Hence, the trajectory $x[t]$ emerges from the point (t^0, x^0) and is contained in Φ up to the instant $\bar{\tau}$ when it enters the target set. This means that $x[t]$ is a viable trajectory of inclusion (1.2). Consequently, $(t^0, x^0) \in \Omega$. By virtue of the arbitrariness of the point $(t^0, x^0) \in \Omega^0$, we obtain $\Omega^0 \subset \Omega$.

Proof of the inclusion $\Omega \subset \Omega^0$. We select an arbitrary point $(t^0, x^0) \in \Omega$ and show that $(t^0, x^0) \in \Omega^0$.

Suppose $y[t]$, $t \in I^0 = [t^0, t^1]$ is a viable trajectory of inclusion (1.2), emerging from the point (t^0, x^0). It satisfies the following conditions

$$y[t] \in \Phi(t), \quad t \in I^0; \quad y[t^1] \in T(t^1); \quad y[t] \notin T(t), \quad t \in [t^0, t^1) \tag{3.10}$$

For each natural n, we denote the instant of the partitioning Γ_n which is closest to t from the right by

$$t_n(t) = \min\{t^{(i)} \in \Gamma_n : t \leq t^{(i)}\}, \quad t \in \mathcal{I} \tag{3.11}$$

We now fix n and consider the interval $[\bar{t}^n, \bar{t}_n]$, $\bar{t}^n = t_n(t^0)$, $\bar{t}_n = t_n(t^1)$. It follows from the definition (3.11) that

$$t^0 \in [\bar{t}^n - \Delta_n, \bar{t}^n]; \quad t^1 \in [\bar{t}_n - \Delta_n, \bar{t}_n] \tag{3.12}$$

Taking into account the second condition of (3.10), the second inclusion of (3.12) and the definition of the set T_n, we obtain

$$y[t^1] \in T_n(\bar{t}_n) \tag{3.13}$$

We now continue the trajectory $y[t]$ on the interval $[t^1, \bar{t}_n]$ in an arbitrary manner but in such a way that it satisfies the inclusion

$$\dot{y}[t] \in F(t, y[t]) \text{ almost everywhere in } [t^1, \bar{t}_n]$$

The trajectory $y[t]$ of inclusion (1.2) is not a viable trajectory in the interval $[t^0, \bar{t}_n]$ but the part of it $y[t]$, $t \in [t^0, t^1]$ is still a viable trajectory.

Using procedure, a backward-stepping, we establish the correspondence of the trajectory $y[t]$, $t \in [t^0, \bar{t}_n]$ to Euler's broken line. We put

$$\tilde{y}_n[\bar{t}_n] = y[t^1] \tag{3.14}$$

and, assuming that the value of $\tilde{y}_n[t^{(i+1)}]$ has already been determined at the instant $t^{(i+1)} \in \Gamma_n$, we construct the vector $f(t^{(i+1)})$ which satisfies the following two conditions

$$f(t^{(i+1)}) \in F(t^{(i+1)}, \tilde{y}_n[t^{(i+1)}])$$

$$\langle f(t^{(i+1)}), s(t^{(i+1)})\rangle = \max\{\langle f, s(t^{(i+1)})\rangle : f \in F(t^{(i+1)}, \tilde{y}_n[t^{(i+1)}])\}$$

where the vector $s(t^{(i+1)}) = y[t^{(i+1)}] - \tilde{y}_n[t^{(i+1)}]$. Here, $\langle\cdot, \cdot\rangle$ is the scalar product of vectors.

Then, as the values of $\tilde{y}_n[t]$, $t \in [t^{(i)}, t^{(i+1)})$, we take

$$\tilde{y}_n[t] = \tilde{y}_n[t^{(i+1)}] - (t - t^{(i+1)})f(t^{(i+1)})$$

Euler's broken line $\tilde{y}_n[t]$ is thereby determined in the interval $[\bar{t}^n, \bar{t}_n]$. By virtue of the definition of this line at the instant $\bar{t}_n$, the following estimate holds

$$\|\tilde{y}_n[\bar{t}_n] - y[\bar{t}_n]\| \le K\Delta_n$$

Furthermore, the estimate

$$\|\tilde{y}_n[t^{(i)}] - y[t^{(i)}]\| \le \varepsilon^{(i)} \tag{3.15}$$

is true for all instants $t^{(i)} \in [\bar{t}^n, \bar{t}_n]$.

The broken line $\tilde{y}_n[t]$, $t \in [\bar{t}^n, \bar{t}_n]$ is shown schematically in Fig. 5.

From relation (3.13) and the definition (3.14), it follows that the broken line $\tilde{y}_n[t]$ reaches the discretization of the target set:

$$\tilde{y}_n[\bar{t}_n] \in T_n(\bar{t}_n) \tag{3.16}$$

Moreover, from inequalities (3.15) and the inclusion $y[t^{(i)}] \in \Phi(t^{(i)})$, we obtain

$$\tilde{y}_n[t^{(i)}] \in \Phi(t^{(i)})_{\varepsilon^{(i)}}, \quad t^{(i)} \in [\bar{t}^n, \bar{t}_n]$$

It follows from this and inclusion (3.16) that

$$\tilde{y}_n[t^{(i)}] \in \Omega_n(t^{(i)}), \quad t^{(i)} \in [\bar{t}^n, \bar{t}_n] \tag{3.17}$$

We now introduce the notation

$$y_n = \tilde{y}_n[\bar{t}^n] \tag{3.18}$$

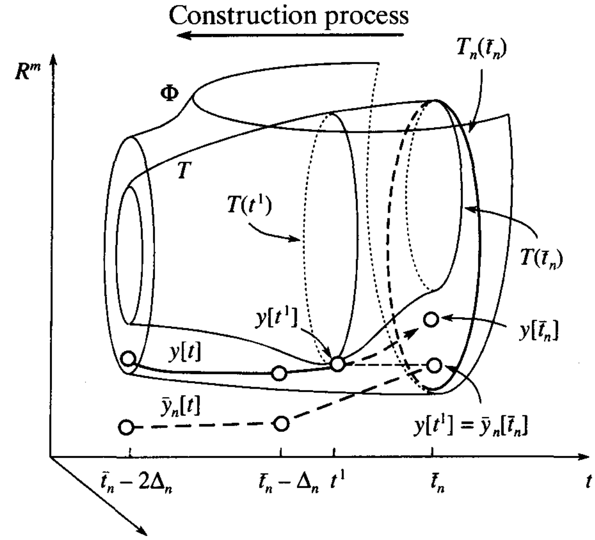

Fig. 5

The sequence $\{(\bar{t}^n, y_n)\}$ converges to the point (t^0, x^0). In fact, it follows from the first inclusion of (3.12) that $\lim \bar{t}^n = t^0$, and we conclude from this that $\lim y[\bar{t}^n] = x^0$. The limit relation

$$\lim y_n = \lim y[\bar{t}^n] = x^0 \tag{3.19}$$

follows from the inequality $\|y_n - y[\bar{t}^n]\| \leq \varepsilon^{(0)}$ (see (3.15)) and Assertion 3.

It therefore follows from relations (3.17)–(3.19) that a sequence $(\bar{t}^n, y_n) \in \Gamma_n \times \Omega_n(\bar{t}^n)$ $(n = 1, 2, \ldots)$ has been found which satisfies the relation

$$(t_0, x_0) = \lim(\bar{t}^n, y_n)$$

Consequently, $(t^0, x^0) \in \Omega^0$. Since the point (t^0, x^0) is arbitrarily chosen in Ω, then $\Omega \subset \Omega^0$.

The theorem is proved.

The proof of the theorem which has been presented does not give upper estimates of the Hausdorff distances $d(\Omega(t^{(i)}), \Omega_n(t^{(i)}))$. It is only possible to indicate a one-side estimate for them. For all natural n and $t^{(i)} \in \Gamma_n$, the following inclusion holds

$$\Omega(t^{(i)}) \subset \Omega_n(t^{(i)})_{\varepsilon^{(i)}}$$

The derivation of this estimate is analogous to the derivation of the estimate from [9].

This research was supported by the Russian Foundation for Basic Research (02-01-00769, 00-15-960571).

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Translated by E.L.S.